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A.Z. Petrov

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JET PROPULSION LABORATORY

CALIFORNIA INSTITUTE OF TECHNOLOGY

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CLASSIFICATION OF SPACES SUPPORTING GRAVITATIONAL FIELDS

A. Z. Petrov

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Translated by Michael Karweit

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CLASSIFICATION OF SPACES SUPPORTING GRAVITATIONAL FIELDS[1]

A. Z. Petrov[2]

This article gives an expanded proof of results derived by the author earlier and first published in 1951 (Ref. 1). We will treat a V_4 supporting a gravitational field (that is, we have in four dimensions

$$ds^2 = g_{ij}\, dx^i\, dx^j \tag{1}$$

and, further, the field equations

$$R_{ij} = \kappa g_{ij} \tag{2}$$

– we will call such a manifold a T_4), and we will establish for it a classification scheme by investigating the algebraic structure of the curvature tensor.

[1] *Scientific Transactions of the Kazan State University*, (Named for V. I. Ul'yanov-Lenin), Jubilee (1804-1954) Collection, Vol. 114, Book 8, 1954, pp. 55-69.

[2] Kazan State University, Department of Geometry.

I. BIVECTOR SPACE

We consider some point P in our manifold T_4 and compare it with a local centro-affine E_4. In this E_4, we single out all the tensors that satisfy the conditions: (1) the number of covariant indices is the same as the number of contravariant indices, and (2) the co- and contravariant indices may be grouped in separate antisymmetrical pairs. We will consider every such pair as one collective index, which we will denote with a Greek letter. In this way, we derive a manifold of $N = n(n-1)/2$ dimensions (six dimensions for $n = 4$). The tensors of E_4 possessing the indicated properties define, in this space, tensors of half their order.

To each point of the T_4, in this way, there corresponds a local six-dimensional centro-affine geometry with a transformation group

$$\begin{aligned} \eta^{\alpha'} &= A^{\alpha'}_{\alpha}\,\eta^{\alpha} \\ \eta^{\alpha} &= A^{\alpha}_{\alpha'}\,\eta^{\alpha'} \\ |A^{\alpha'}_{\alpha}| &\neq 0 \\ A^{\alpha}_{\beta}\,A^{\beta'}_{\gamma} &= \delta^{\alpha}_{\gamma} \end{aligned} \tag{3}$$

In fact, if we order the collective index (choosing one from the two possible pairs ij and ji), then we have six possible collective indices. We may establish, for example, the following correspondence:

1 - 14	2 - 24	3 - 34
4 - 23	5 - 31	6 - 12

We consider now the transformation of the components T^{ij} of a general bivector:

$$T^{i'j'} = A^{i'j'}_{ij}\,T^{ij}$$

Setting

$$A^{\alpha'}_{\alpha} = 2A^{[i'j']}_{ij} \qquad \left[\text{where } A^{i'}_{i} = \left(\frac{\partial x^{i'}}{\partial x^{i}}\right)_P\right]$$

we readily derive in terms of the collective indices the relationship

$$T^{\alpha'} = A_{\alpha}^{\alpha'} T^{\alpha}$$

that is, the set of the bivectors of T_n (the dimension here is not significant) determines in E_N a set of contravariant vectors satisfying the conditions of Eq. (3). These relationships may be verified immediately by a change to Latin indices.

We will call the derived manifold a bivector manifold. In what follows, the curvature tensor of T_4 will be of special interest; in bivector space, it will correspond to a symmetrical tensor of rank two:

$$R_{ijkl} \to R_{\alpha\beta} = R_{\beta\alpha}$$

In each local E_6, one may introduce a metric, using for this purpose some tensor of T_4 possessing the properties

$$M_{ijkl} = M_{klij} = -M_{jikl} = -M_{ijlk}$$

and with the condition that the rank two tensor in E_6 corresponding to it is nonsingular. As such a fundamental tensor of E_6 we take the tensor

$$g_{ikjl} = g_{ij}\, g_{kl} - g_{il}\, g_{kj} \to g_{\alpha\beta} = g_{\beta\alpha} \qquad (4)$$

$$ik \to \alpha \qquad\qquad jl \to \beta$$

It is easy to see that $g_{\alpha\beta}$ gives a nondegenerate metric, since $|g_{ij}| \neq 0$, and

$$|g_{\alpha\beta}| = p\, |g_{ij}|^{2n} \qquad\qquad p \neq 0$$

For a determinate g_{ij}, $g_{\alpha\beta}$ also will be determinate; for an indeterminate g_{ij}, the tensor $g_{\alpha\beta}$ also will, in general, be indeterminate. We note that we will consider only those gravitational fields that correspond to a real distribution of matter in space; for this it is necessary (Ref. 2) that, at every given point of T_4, the fundamental tensor g_{ij} in a real system of coordinates be reducible to the form

$$(g_{ij}) = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \qquad (5)$$

that is, we arrive, in such a way, at Minkowski space. Then from Eq. (4), it follows that, for the frame of reference corresponding to the matrix (5), the fundamental tensor of R_6 will have the form

$$(g_{\alpha\beta}) = \begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \qquad |g_{\alpha\beta}| = -1 \tag{6}$$

that is, $g_{\alpha\beta}$ is an indeterminate tensor.

II. CLASSIFICATION OF T_4

A series of most interesting problems arising in the study of Riemannian manifolds is connected with the curvature tensor of V_n. With the help of this tensor, as is known, the curvature may be evaluated in a given two-dimensional direction at a given point:

$$K = \frac{R_{ijkl} V^{ij} V^{kl}}{g_{pqrs} V^{pq} V^{rs}} \tag{7}$$

where g_{pqrs} takes the form of Eq. (4), and where the two-space direction determined by $\underset{1}{V^i}$, $\underset{2}{V^i}$ is characterized by the simple bivector $V^{ij} = \underset{[1}{V^i} \underset{2]}{V^j}$. We now introduce the general curvature of V_n, by relaxing in Eq. (7) the requirement of simplicity of the bivector V^{ij}. The general invariant K at some point in V_n will be a homogeneous function of the components of the bivector V^{ij} (not simple, in general, and of zero rank); evidently, it may be written in the form

$$K = \frac{R_{\alpha\beta} V^{\alpha} V^{\beta}}{g_{\alpha\beta} V^{\alpha} V^{\beta}} \tag{8}$$

We consider now the problem of determining the critical values of K, which is equivalent to finding the vectors V^{α} in R_N for which K assumes the critical values. Let us agree to call these critical values of K *stationary curvatures* of V_n and to call the corresponding bivectors V^{α} the *stationary directions* of V_n. In this way, the problem is reduced to the determination of *unconditional-stationary vectors* of V^{α} in bivector space, based on the necessary and sufficient conditions of stationariness:

$$\frac{\partial K}{\partial V^{\alpha}} = 0 \tag{9}$$

It is necessary to bear in mind that, with an indeterminate g_{ij} tensor, $g_{\alpha\beta}$ is also indeterminate; consequently, the appearance of null stationary directions is possible:

$$g_{\alpha\beta} V^{\alpha} V^{\beta} = 0 \tag{10}$$

For the time being, we exclude this particular case, but will return to it below.

If Eq. (10) does not take place, then Eq. (9) reduces to

$$(R_{\alpha\beta} - K g_{\alpha\beta}) V^{\beta} = 0 \tag{11}$$

that is, the stationary directions of V_n will be the principal directions of the tensor $R_{\alpha\beta}$ in bivector space, and the stationary curvatures of V_n will be the characteristic quantities of the secular equation

$$|R_{\alpha\beta} - K g_{\alpha\beta}| = 0 \tag{12}$$

Let Eq. (10) hold for a stationary V^{α}. Since we are interested in K only, satisfying Eq. (9), K is a continuous function of V^{α}; consequently, it is necessary that the condition $R_{\alpha\beta}V^{\alpha}V^{\beta} = 0$ be fulfilled. Then the value of K for the stationary null direction of V^{α} may be calculated in the following way:

$$K(V^{\alpha}) = \lim_{dV^{\alpha} \to 0} K(V^{\alpha} + dV^{\alpha})$$

If we designate, for arbitrary V^{α},

$$\phi = g_{\alpha\beta} V^{\alpha} V^{\beta} \qquad\qquad \psi = R_{\alpha\beta} V^{\alpha} V^{\beta} \tag{13}$$

then, for a stationary null V^{α},

$$K(V^{\alpha}) = \lim_{dV^{\alpha} \to 0} \frac{\psi(V^{\alpha} + dV^{\alpha}) - \psi(V^{\alpha})}{\phi(V^{\alpha} + dV^{\alpha}) - \phi(V^{\alpha})}$$

$$= \lim \frac{\sum_{\sigma} \dfrac{\partial}{\partial V^{\sigma}} \psi \, dV^{\sigma} + \cdots}{\sum_{\sigma} \dfrac{\partial}{\partial V^{\sigma}} \phi \, dV^{\sigma} + \cdots}$$

and, since this limit may not depend on the method of variation of dV^α,

$$K(V^\alpha) = \frac{\dfrac{\partial}{\partial V^\sigma}\psi}{\dfrac{\partial}{\partial V^\sigma}\phi} = \frac{R_{\alpha\beta}V^\beta}{g_{\alpha\beta}V^\beta}$$

that is, we again arrive at Eq. (11).

The determination of stationary curves and directions of R_N leads to the study of the pair of quadratic forms given in Eq. (13). Consequently, canonical expression of this pair of forms in real space permits classification of the curvature tensor of V_n not only at the given point, but also in the region of V_n that includes this point, in which the characteristic K-matrix

$$||R_{\alpha\beta} - Kg_{\alpha\beta}|| \tag{14}$$

is invariant. To every type of characteristic matrix (14), there corresponds a special kind of gravitational field that determines the desired classification of T_4.

By using a real transformation, the matrix $||g_{\alpha\beta}||$ may be converted to the form of Eq. (6); by using real orthogonal transformations, the matrix $||R_{\alpha\beta}||$ may be simplified.

Theorem 1. The matrix $||R_{\alpha\beta}||$ in an orthogonal frame of reference (Eq. 5) may be symmetrically factored.

In a frame of reference (Eq. 5), the equation of the field assumes the form

$$\sum_k e_k R_{ikjk} = \kappa g_{ij} \qquad e_k = \pm 1$$

that is, with $i = j$,

$$\sum_k e_k R_{ikik} = \kappa e_i$$

and, with $i \neq j$,

$$e_k R_{ikjk} + e_l R_{iljl} = 0 \qquad (i, j, k, l \neq)$$

Describing these relationships in the selected indices of bivector space and taking into account the numeration introduced in Part I, we derive for the matrix the expression

$$\|R_{\alpha\beta}\| = \left\| \begin{array}{c|c} M & N \\ \hline N & -M \end{array} \right\| \tag{15}$$

where

$$M = \begin{Vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{Vmatrix}$$

$$N = \begin{Vmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{Vmatrix}$$

$$\begin{aligned} m_{\alpha\beta} &= m_{\beta\alpha} \\ n_{\alpha\beta} &= n_{\beta\alpha} \end{aligned} \qquad (\alpha, \beta = 1, 2, 3)$$

and where

$$\sum_{i=1}^{3} m_{ii} = \kappa \qquad \text{and} \qquad \sum_{1}^{3} n_{ii} = 0$$

because of the Ricci identity. We note that V. F. Kagan arrived at this kind of matrix, however, with the additional condition of orthogonality, by studying a group of Lorentz transformations (Ref. 3). Y. S. Dubnov (Ref. 4) and A. M. Lopshitz (Ref. 5) studied this type of matrix with the same assumption of orthogonality. The fact proved in Theorem 1 holds for any orthogonal frame of reference, and, consequently, considering that an orthogonal frame of reference is determined for $n = 4$ up to six degrees of freedom, one may further simplify the matrix by a choice of six rotations.

First we prove a theorem that sharply restricts the number (at first glance) of types of characteristic matrices (14).

Theorem 2. The characteristic K-matrix (14) always consists of two equal parts.

We reduce matrix (14) to a simpler form by using so-called elementary transformations that, as is known, change neither the elementary divisors of the matrix nor, consequently, its characteristic. We represent this matrix as

$$\left\| \begin{array}{c|c} m_{\alpha\beta} + K\,\delta_{\alpha\beta} & n_{\alpha\beta} \\ \hline n_{\alpha\beta} & -m_{\alpha\beta} - K\,\delta_{\alpha\beta} \end{array} \right\|$$

where $\delta_{\alpha\beta}$ is the Kronecker delta.

Adding to each of the three first columns the corresponding column of the second three, multiplied by i, we derive the equivalent matrix

$$\left\| \begin{array}{c|c} m_{\alpha\beta} + i\,n_{\alpha\beta} + K\,\delta_{\alpha\beta} & n_{\alpha\beta} \\ \hline -i(m_{\alpha\beta} + i\,n_{\alpha\beta} + K\,\delta_{\alpha\beta}) & -m_{\alpha\beta} - K\,\delta_{\alpha\beta} \end{array} \right\|$$

Adding to each of the second three rows the corresponding row of the first three, multiplied by i, we obtain

$$\left\| \begin{array}{c|c} m_{\alpha\beta} + i\,n_{\alpha\beta} + K\,\delta_{\alpha\beta} & n_{\alpha\beta} \\ \hline 0 & -m_{\alpha\beta} + i\,n_{\alpha\beta} - K\,\delta_{\alpha\beta} \end{array} \right\|$$

Finally, multiplying the first three columns by $i/2$ and adding to the corresponding second of three columns, and then doing the same with the second three rows, we reduce the matrix to

$$\left\| \begin{array}{c|c} m_{\alpha\beta} + i\,n_{\alpha\beta} + K\,\delta_{\alpha\beta} & 0 \\ \hline 0 & m_{\alpha\beta} - i\,n_{\alpha\beta} + K\,\delta_{\alpha\beta} \end{array} \right\| \equiv \left\| \begin{array}{cc} P(K) & 0 \\ \hline 0 & \overline{P}(K) \end{array} \right\|$$

equivalent to the K-matrix (14). The problem is thus reduced to study of two three-dimensional matrices $P(K)$ and $\overline{P}(K)$, the corresponding elements of which are complex-conjugate. From here it follows that the elementary divisors of these two matrices are also complex-conjugate, and, consequently, their characteristics have identical forms. In this way, the characteristic of our K-matrix divides into two similar parts – which is correct.

We note that the principal directions and invariant groups of the K-matrix also should be complex-conjugate in pairs.

Now one may introduce the classification of gravitational fields:

THEOREM 3. There exist three and only three types of gravitational field. The three-dimensional matrix $P(K)$ may have only one of three possible types of characteristic: $[1\ 1\ 1]$, $[2\ 1]$, $[3]$, if one disregards the case in which some of the elementary divisors have identical bases and, consequently, some of the numbers appearing in the brackets are included in round parentheses (for example: $[(11)\ 1]$, $[(21)]$, etc.).

The characteristic of $\overline{P}(K)$ may take the same form. Then the characteristics of the K-matrix will be written

$$\begin{aligned} &1.\ [1\,\bar{1},\ 1\,\bar{1},\ 1\,\bar{1}] \\ &2.\ [2\,\bar{2},\ 1\,\bar{1}] \\ &3.\ [3\,\bar{3}] \end{aligned} \tag{16}$$

where in each case the dashed numbers designate the power indices for the elementary divisor with a base that is complex-conjugate to the base of the elementary divisor, the power of which is expressed by the preceding number.

Each type of gravitational field will be studied separately in Part III; for each type we will derive the canonical forms of the matrix $\|R_{\alpha\beta}\|$.

III. THE CANONICAL FORMS OF THE MATRIX $||R_{\alpha\beta}||$

A. Manifold T_4 with Characteristic $[1\bar{1},\ 1\bar{1},\ 1\bar{1}]$

We consider the first type of T_4 with characteristic $[1\bar{1},\ 1\bar{1},\ 1\bar{1}]$. Since in this case the characteristic is of the simple type, the tensor R has six nonnull, mutually orthogonal principal directions (Ref. 6). These directions of bivector space at the given point T_4 will give the bivectors a specific structure, which will be derived.

We denote the component vectors of a real orthogonal frame of reference at a point of T_4 by $\underset{k}{\xi}{}^i$ $(k, i = 1, 2, 3, 4)$; the simple bivector $\underset{[k}{\xi}{}^i \underset{l]}{\xi}{}^j$ $(k \neq l)$, distributing the two-space (area) determined by the frame of reference, will be, for shortness, designated ξ^{ij}_{kl}. In bivector space these simple bivectors determine six independent, nonnull, mutually orthogonal coordinate vectors $\underset{\sigma}{\xi}{}^\alpha = \delta^\alpha_\sigma$; any vector of R_6, in particular the vectors of the principal directions of $R_{\alpha\beta}$, may be expressed in terms of these vectors.

We will show that, for the vectors of the principal directions [they are determined uniquely only when the roots of the secular equation (Eq. 12) are all different], one may take the vectors in the form

$$W^\alpha = \lambda\left(\underset{1}{\xi}{}^\alpha \pm i\underset{4}{\xi}{}^\alpha\right) + \mu\left(\underset{2}{\xi}{}^\alpha \pm i\underset{5}{\xi}{}^\alpha\right) + \nu\left(\underset{3}{\xi}{}^\alpha \pm i\underset{6}{\xi}{}^\alpha\right) \tag{17}$$

In most cases, the condition that W^α determines the principal direction of the tensor $R_{\alpha\beta}$ is written

$$(R_{\alpha\beta} - K g_{\alpha\beta}) W^\beta = 0 \tag{18}$$

But this system of six equations, because of the symmetry of the K-matrix, reduces to three equations:

$$(m_{s1} \pm i n_{s1} + K)\lambda + (m_{s2} \pm i n_{s2})\mu + (m_{s3} \pm i n_{s3})\nu = 0 \qquad (s = 1, 2, 3)$$

In order that λ, μ, ν be nonzero solutions of this system, it is necessary and sufficient that K be a root of one of the equations

$$|P(K)| = 0 \qquad |\bar{P}(K)| = 0 \tag{19}$$

that is, a root of the secular equation (Eq. 12); this proves the form of Eq. (17).

To the vector W (Eq. 17) of manifold R_6 at a given point of T_4, there corresponds the bivector

$$W^{ij} = \lambda\left(\underset{14}{\xi}^{ij} \pm i\, \underset{23}{\xi}^{ij}\right) + \mu\left(\underset{24}{\xi}^{ij} \pm i\, \underset{31}{\xi}^{ij}\right) + \nu\left(\underset{34}{\xi}^{ij} \pm i\, \underset{12}{\xi}^{ij}\right) \tag{20}$$

It is not difficult to see that, under a real orthogonal transformation, W^{ij} transforms into a bivector of the same type, during which $\lambda, \mu, \nu \to \overset{*}{\lambda}, \overset{*}{\mu}, \overset{*}{\nu}$, so that the norm of the bivector remains invariant:

$$\lambda^2 + \mu^2 + \nu^2 = \overset{*}{\lambda}{}^2 + \overset{*}{\mu}{}^2 + \overset{*}{\nu}{}^2$$

Let the roots of (Eq. 12), $\underset{s}{K}(s = 1, 2, 3)$, correspond to the vectors of principal direction $\underset{s}{W}^{\alpha}$; then the $\underset{s+3}{K}$, according to the preceding, must correspond to $\underset{s}{\overline{W}}^{\alpha}$, with proper numeration. To the root $\underset{1}{K}$ corresponds the bivector

$$\underset{1}{W}^{pq} = \underset{1}{\lambda}\left(\underset{14}{\xi}^{pq} + i\, \underset{23}{\xi}^{pq}\right) + \underset{1}{\mu}\left(\underset{24}{\xi}^{pq} + i\, \underset{31}{\xi}^{pq}\right) + \underset{1}{\nu}\left(\underset{34}{\xi}^{pq} + i\, \underset{12}{\xi}^{pq}\right)$$

and to $\underset{4}{K}$ – the bivector

$$\underset{4}{W}^{pq} = \underset{1}{\overline{\lambda}}\left(\underset{14}{\xi}^{pq} - i\, \underset{23}{\xi}^{pq}\right) + \underset{1}{\overline{\mu}}\left(\underset{24}{\xi}^{pq} - i\, \underset{31}{\xi}^{pq}\right) + \underset{1}{\overline{\nu}}\left(\underset{34}{\xi}^{pq} - i\, \underset{12}{\xi}^{pq}\right)$$

We represent the bivector $\underset{1}{W}^{pq}$ as a sum of two real bivectors $\underset{1}{V}^{pq} + i\underset{1}{\overset{*}{V}}{}^{pq}$; then,

$$\underset{4}{W}^{pq} = \underset{1}{V}^{pq} - i\, \underset{1}{\overset{*}{V}}{}^{pq}$$

Let

$$\lambda = \underset{1}{a} + i\, \underset{1}{b}$$

$$\mu = \underset{2}{a} + i\, \underset{2}{b}$$

$$\nu = \underset{3}{a} + i\, \underset{3}{b}$$

where $\underset{s}{a}, \underset{s}{b}$ are real numbers $(s = 1, 2, 3)$ and, consequently,

$$\underset{1}{V}^{pq} = \underset{1}{a}\,\underset{14}{\xi}^{pq} + \underset{2}{a}\,\underset{24}{\xi}^{pq} + \underset{3}{a}\,\underset{34}{\xi}^{pq} - \underset{1}{b}\,\underset{23}{\xi}^{pq} - \underset{2}{b}\,\underset{31}{\xi}^{pq} - \underset{3}{b}\,\underset{12}{\xi}^{pq}$$

$$\overset{*}{\underset{1}{V}}{}^{pq} = \underset{1}{b}\,\underset{14}{\xi}^{pq} + \underset{2}{b}\,\underset{24}{\xi}^{pq} + \underset{3}{b}\,\underset{34}{\xi}^{pq} + \underset{1}{a}\,\underset{23}{\xi}^{pq} + \underset{2}{a}\,\underset{31}{\xi}^{pq} + \underset{3}{a}\,\underset{12}{\xi}^{pq}$$

Since $\underset{1}{W}^{\alpha}$ is a nonnull vector of R_6, one may consider it to be a unit vector

$$g_{\alpha\beta}\,\underset{1}{W}^{\alpha}\,\underset{1}{W}^{\beta} = 1$$

From this we come to the conclusion that

$$\sum_{s=1}^{3} \underset{s}{a}\,\underset{s}{b} = 0 \tag{21}$$

$$\sum_{s=1}^{3} (\underset{s}{b}^2 - \underset{s}{a}^2) > 1 \tag{22}$$

Now one may verify the following statements:

1. The real bivectors $\underset{1}{V}^{pq}$ and $\overset{*}{\underset{2}{V}}{}^{pq}$ are *one-sided.*
2. The real bivectors $\underset{1}{V}^{pq}$ and $\overset{*}{\underset{2}{V}}{}^{pq}$ are *O-parallel* (from Eq. 21). They may not be 2/2-parallel since that may happen only under the condition that the coefficients of identical $\underset{ij}{\xi}^{pq}$ are proportional; in our case, they would have to be zero, since if

$$\frac{\underset{1}{a}}{\underset{1}{b}} = -\frac{\underset{1}{b}}{\underset{1}{a}}$$

then

$$\underset{1}{a}^2 + \underset{1}{b}^2 = 0$$

They may not be 1/2-parallel since then $\underset{1}{W}^{\alpha}$ would be a single-sided complex bivector, but, using the condition of simplicity, we could arrive at a contradiction between Eq. (21) and (22). Thus there remains only the possibility indicated above.

3. The real bivectors $\underset{1}{V}^{pq}$ and $\underset{2}{\overset{*}{V}}{}^{pq}$ are 2/2-*perpendicular.* For this it is necessary and sufficient that, with arbitrary i and j, the equality be fulfilled:

$$\underset{1}{V}_{is}\,\overset{*}{V}{}^{sj} = 0$$

It is easy to see that this reduces to Eq. (21) and, consequently, is true.

We consider now the simple bivector $\underset{1}{V}^{pq}$. Its norm, from Eq. (22),is

$$g_{\alpha\beta}\,\underset{1}{V}^{\alpha}\underset{1}{V}^{\beta} = \Sigma\,(\underset{s}{b}^2 - \underset{s}{a}^2) > 0$$

In the plane of this real bivector, one may select two real, orthogonal, nonnull vectors η^p, ν^p. Then the norm of our bivector may be expressed in the form

$$2\,\eta_p\,\eta^p \cdot \nu_q\,\nu^q$$

and, consequently, these two vectors are either both space-like or both time-like. Their norms may not be greater than zero since, assuming these two orthogonal real vectors for coordinates, we contradict the law of inertia of the quadratic form. Consequently, these two vectors both have negative norms. In view of this, by renormalizing, one may take them as the vectors $\underset{2}{\overset{*}{\xi}}{}^i$, $\underset{3}{\overset{*}{\xi}}{}^i$ of a new real orthogonal frame of reference.

In exactly the same way, in the plane of $\underset{1}{\overset{*}{V}}{}^{pq}$ we determine two orthogonal vectors, real and nonnull, but having norms of opposite signs such that

$$g_{\alpha\beta}\,\underset{1}{\overset{*}{V}}{}^{\alpha}\underset{1}{\overset{*}{V}}{}^{\beta} < 0$$

We call these vectors $\underset{1}{\overset{*}{\xi}}{}^i$, $\underset{4}{\overset{*}{\xi}}{}^i$. In this system of coordinates

$$\underset{1}{\overset{*}{W}}{}^{pq} = \underset{14}{\xi}^{pq} + i\,\underset{23}{\xi}^{pq}$$

$$\underset{4}{\overset{*}{W}}{}^{pq} = \underset{14}{\xi}^{pq} - i\,\underset{23}{\xi}^{pq}$$

We note that the frame of reference $\overset{*}{\xi}$ is chosen up to a rotation in the plane of $\left\{\underset{2}{\overset{*}{\xi}}\,\underset{3}{\overset{*}{\xi}}\right\}$ and up to a Lorentz rotation in the plane of $\left\{\underset{1}{\overset{*}{\xi}}\,\underset{4}{\overset{*}{\xi}}\right\}$ The bivectors W^{pq} are of interest, of course, only up to a scalar factor.

Now, writing the condition of orthogonality of $\underset{1}{\overset{*}{W}}{}^{pq}$ and $\underset{2}{\overset{*}{W}}{}^{pq}$, we find that the bivector of the second principal direction must take the form

$$\underset{2}{\overset{*}{W}}{}^{pq} = \underset{2}{\overset{*}{\mu}}\left(\underset{24}{\overset{*}{\xi}}{}^{pq} + i\,\underset{31}{\overset{*}{\xi}}{}^{pq}\right) + \underset{2}{\overset{*}{\nu}}\left(\underset{34}{\overset{*}{\xi}}{}^{pq} + i\,\underset{12}{\overset{*}{\xi}}{}^{pq}\right)$$

We use the frame of reference indicated above and perform the rotation

$$\underset{1}{\xi}{}^{p} = \cosh\phi\,\underset{1}{\overset{*}{\xi}}{}^{p} + \sinh\phi\,\underset{4}{\overset{*}{\xi}}{}^{p}$$

$$\underset{4}{\xi}{}^{p} = \sinh\phi\,\underset{1}{\overset{*}{\xi}}{}^{p} + \cosh\phi\,\underset{4}{\overset{*}{\xi}}{}^{p}$$

$$\underset{2}{\xi}{}^{p} = \cos\psi\,\underset{2}{\overset{*}{\xi}}{}^{p} + \sin\psi\,\underset{3}{\overset{*}{\xi}}{}^{p}$$

$$\underset{3}{\xi}{}^{p} = -\sin\psi\,\underset{2}{\overset{*}{\xi}}{}^{p} + \cos\psi\,\underset{3}{\overset{*}{\xi}}{}^{p}$$

After these transformations, $\underset{1}{W}$ will take the same form, and $\underset{2}{W}$ becomes

$$\underset{2}{\tilde{W}}{}^{pq} = \underset{2}{\tilde{\mu}}\left(\underset{24}{\tilde{\xi}}{}^{pq} + i\,\underset{31}{\tilde{\xi}}{}^{pq}\right) + \underset{2}{\tilde{\nu}}\left(\underset{34}{\tilde{\xi}}{}^{pq} + i\,\underset{12}{\tilde{\xi}}{}^{pq}\right)$$

where

$$\underset{2}{\tilde{\nu}} = \sin\psi\cosh\phi + p\cos\psi\cosh\phi + q\sin\psi\sinh\phi$$
$$+\, i\,(\cos\psi\sinh\phi + q\cos\psi\cosh\phi - p\sin\psi\sinh\phi)$$

$$p + iq = \frac{\underset{2}{\overset{*}{\nu}}}{\underset{2}{\overset{*}{\mu}}}$$

and where $\underset{2}{\overset{*}{\mu}}$ may be considered different from zero, since otherwise we would have $\phi = \psi = 0$. One can find a real ϕ and ψ for every $\underset{2}{\tilde{\nu}} = 0$. Now the frame of reference is uniquely determined, and in it, taking account of the orthogonality of $\underset{1}{W}$, $\underset{2}{W}$, and $\underset{3}{W}$, these bivectors take the form (within a scalar factor)

$$\underset{1}{W}^{pq} = \underset{14}{\xi}^{pq} + i\,\underset{23}{\xi}^{pq}$$

$$\underset{2}{W}^{pq} = \underset{24}{\xi}^{pq} + i\,\underset{31}{\xi}^{pq}$$

$$\underset{3}{W}^{pq} = \underset{34}{\xi}^{pq} + i\,\underset{12}{\xi}^{pq}$$

Because of the complex conjugation indicated above,

$$\underset{4}{W}^{pq} = \underset{1}{\overline{W}}^{pq}$$

$$\underset{5}{W}^{pq} = \underset{2}{\overline{W}}^{pq}$$

$$\underset{6}{W}^{pq} = \underset{3}{\overline{W}}^{pq}$$

Now, writing condition (18) for each of these bivectors and taking

$$\underset{\alpha}{\xi}^{\sigma} = \delta_{\alpha}^{\sigma}$$

we easily find

$$\left.\begin{aligned} m_{ii} &= \underset{i}{\alpha} \\ m_{ij} &= 0 \\ n_{ii} &= -\underset{i}{\beta} \\ n_{ij} &= 0 \end{aligned}\right\} \quad (i = 1,\ 2,\ 3;\ i \neq j)$$

and, consequently, for the first type of T_4, we have derived the following canonical matrix:

$$(R_{\alpha\beta}) = \left\| \begin{array}{ccc|ccc} -\alpha_1 & & & -\beta_1 & & \\ & -\alpha_2 & & & -\beta_2 & \\ & & -\alpha_3 & & & -\beta_3 \\ \hline -\beta_1 & & & \alpha_1 & & \\ & -\beta_2 & & & \alpha_2 & \\ & & -\beta_3 & & & \alpha_3 \end{array} \right\| \tag{23}$$

where the real parts of the stationary curvatures are connected by the relationship

$$\sum_1^3 \alpha_s = \kappa \tag{24}$$

and the imaginary parts, because of the Ricci identity

$$R_{1423} + R_{1234} + R_{1342} = 0$$

satisfy

$$\sum_1^3 \beta_s = 0 \tag{25}$$

B. Manifold T_4 with Characteristic $[21, \overline{2}\overline{1}]$

We now discuss a T_4 with a characteristic of the second type: $[21, \overline{2}\overline{1}]$. As was indicated in Part II, for the principal directions and invariants of a group of K matrices, one may take the principal directions and invariants of the group of matrices $P(K)$ and $\overline{P}(K)$. It follows that it is sufficient to study the matrix $P(K)$ having the characteristic $[21]$.

With such characteristic, the tensor $P_{\alpha\beta} = -m_{\alpha\beta} + i\, n_{\alpha\beta}$ of three-space has (Ref. 6) one principal nonnull direction

$$\left(P_{\alpha\beta} - \underset{1}{K}\, \delta_{\alpha\beta}\right)\underset{1}{W}^{\beta} = 0 \tag{26}$$

Orthogonal to $\underset{1}{W}$ there is a null principal direction $\underset{2}{W}$

$$\left(P_{\alpha\beta} - \underset{2}{K}\, \delta_{\alpha\beta}\right)\underset{2}{W}^{\beta} = 0 \tag{27}$$

Besides there exists a null-vector $\underset{3}{W}^{\beta}$, orthogonal to $\underset{1}{W}^{\beta}$, and nonorthogonal to $\underset{2}{W}^{\beta}$, which together with these vectors forms an invariant square $\left\{\underset{2}{W}, \underset{3}{W}\right\}$ of the tensor $P_{\alpha\beta}$, which is expressed by the relation

$$\left(P_{\alpha\beta} - \underset{2}{K}\, \delta_{\alpha\beta}\right)\underset{3}{W}^{\beta} = \sigma \underset{2}{W}_{\alpha} \tag{28}$$

where σ is an arbitrary scalar different from zero. This arbitrariness is the result of $\underset{2}{W}$ and $\underset{3}{W}$ being null.

Each principal direction and group of $P_{\alpha\beta}$ will determine corresponding principal directions and groups of $R_{\alpha\beta}$; they will all be defined by bivectors of the type in Eq. (17).

Let the root $\underset{1}{K}$ correspond to a simple elementary divisor $(K - \underset{1}{K})$ of a field of K-matrices, and let the principal direction be determined by the bivector $\underset{1}{W}^{\alpha}$. Since this bivector is nonnull, all the arguments set forth for $\underset{1}{W}^{\alpha}$ in the preceding case apply to it. Consequently, one may choose a real frame of reference in which

$$\underset{1}{W}^{pq} = \underset{14}{\xi}^{pq} + i\, \underset{23}{\xi}^{pq}$$

This frame of reference is determined up to a rotation in the plane $\left\{\overline{\underset{2}{\xi}}\, \overline{\underset{3}{\xi}}\right\}$ and a Lorentz rotation in the plane $\left\{\overline{\underset{1}{\xi}}\, \overline{\underset{4}{\xi}}\right\}$. Since the bivectors $\underset{2}{W}^{pq}$ and $\underset{3}{W}^{pq}$ must be orthogonal to $\underset{1}{W}^{pq}$, they take the form

$$\underset{2}{W}^{pq} = \underset{2}{\mu}\left(\underset{24}{\xi}^{pq} + i\, \underset{31}{\xi}^{pq}\right) + \underset{2}{\nu}\left(\underset{34}{\xi}^{pq} + i\, \underset{12}{\xi}^{pq}\right)$$

$$\underset{3}{W}^{pq} = \underset{3}{\mu}\left(\underset{24}{\xi}^{pq} + i\, \underset{31}{\xi}^{pq}\right) + \underset{3}{\nu}\left(\underset{34}{\xi}^{pq} + i\, \underset{12}{\xi}^{pq}\right)$$

The condition of nullness of these bivectors requires

$$\underset{2}{\mu}^2 + \underset{2}{\nu}^2 = 0$$

$$\underset{3}{\mu}^2 + \underset{3}{\nu}^2 = 0$$

that is,

$$\underset{2}{\nu} = e_1 i \underset{2}{\mu}$$

$$\underset{3}{\nu} = e_2 i \underset{3}{\mu}$$

where e_1 and e_2 are ± 1. Finally, considering the fact that they may not be orthogonal, we find that $e_1 = e_2$. Consequently, one may assume

$$\underset{2}{\Psi}^{pq} = \underset{24}{\xi}^{pq} + i \underset{31}{\xi}^{pq} + i\left(\underset{34}{\xi}^{pq} + i \underset{12}{\xi}^{pq}\right)$$

$$\underset{3}{\Psi}^{pq} = \lambda\left\{\underset{24}{\xi}^{pq} + i \underset{31}{\xi}^{pq} - i\left(\underset{34}{\xi}^{pq} + i \underset{12}{\xi}^{pq}\right)\right\}$$

where λ is an arbitrary scalar multiplier other than zero.

Now it remains for us to describe the conditions analogous to the conditions of Eq. (26), (27), and (28) for the tensor $R_{\alpha\beta}$, again taking $\underset{\nu}{\xi}^{\alpha} = \delta^{\alpha}_{\nu}$. These conditions will have the form

$$\left(R_{\alpha\beta} - \underset{1}{K} g_{\alpha\beta}\right) \underset{1}{\Psi}^{\beta} = 0$$

$$\left(R_{\alpha\beta} - \underset{2}{K} g_{\alpha\beta}\right) \underset{2}{\Psi}^{\beta} = 0$$

$$\left(R_{\alpha\beta} - K g_{\alpha\beta}\right) \underset{3}{\Psi}^{\beta} = \sigma g_{\alpha\beta} \underset{2}{\Psi}^{\beta}$$

The tensor $g_{\alpha\beta}$ is determined by the matrix (6). Taking in turn $\alpha = 1, 2, \cdots, 6$, we easily find that the matrix $(R_{\alpha\beta})$ (Eq. 11) will have the form

$$R_{\alpha\beta} = \left\| \begin{array}{ccc|ccc} -\underset{1}{\alpha} & 0 & 0 & -\underset{1}{\beta} & 0 & 0 \\ 0 & -\underset{2}{\alpha} + \sigma & 0 & 0 & -\underset{2}{\beta} & \sigma \\ 0 & 0 & -\underset{2}{\alpha} - \sigma & 0 & \sigma & -\underset{2}{\beta} \\ \hline -\underset{1}{\beta} & 0 & 0 & \underset{1}{\alpha} & 0 & 0 \\ 0 & -\underset{2}{\beta} & \sigma & 0 & \underset{2}{\alpha} - \sigma & 0 \\ 0 & \sigma & -\underset{2}{\beta} & 0 & 0 & \underset{2}{\alpha} + \sigma \end{array} \right\| \qquad \sigma \neq 0 \tag{29}$$

Here σ may be chosen arbitrarily but not equal to zero; $\underset{s}{\alpha}$ and $\underset{s}{\beta}$ as before are related:

$$\begin{aligned} \underset{1}{\alpha} + 2\,\underset{2}{\alpha} &= \kappa \\ \underset{1}{\beta} + 2\,\underset{2}{\beta} &= 0 \end{aligned} \tag{30}$$

The frame of reference is determined up to a rotation in the plane $\left\{\underset{2}{\xi}, \underset{3}{\bar{\xi}}\right\}$ and a Lorentz rotation in the plane $\left\{\underset{1}{\bar{\xi}}\, \underset{4}{\bar{\xi}}\right\}$.

C. Manifold T_4 with Characteristic $[3, \bar{3}]$

The third type of T_4 with characteristic $[3, \bar{3}]$ remains to be considered. With such a characteristic (Ref. 6) for the tensor $P_{\alpha\beta}$, we find one principal nonnull direction $\underset{1}{W}^{\beta}$ and, in addition, two bivectors $\underset{2}{W}^{\beta}$ and $\underset{3}{W}^{\beta}$ possessing the properties

$$\left(P_{\alpha\beta} - \underset{1}{K}\,\delta_{\alpha\beta}\right)\underset{1}{W}^{\beta} = 0$$

$$\left(P_{\alpha\beta} - \underset{1}{K}\,\delta_{\alpha\beta}\right)\underset{2}{W}^{\beta} = \sigma\,\delta_{\alpha\beta}\,\underset{1}{W}^{\beta} \tag{31}$$

$$\left(P_{\alpha\beta} - \underset{1}{K}\,\delta_{\alpha\beta}\right)\underset{3}{W}^{\beta} = \tau\,\delta_{\alpha\beta}\,\underset{2}{W}^{\beta}$$

where σ and τ are arbitrary numbers not equal to zero. The vector $\underset{2}{W}^{\alpha}$ is nonnull, but $\underset{3}{W}^{\alpha}$ is null. In addition, $\underset{1}{W}^{\alpha}$ is orthogonal to $\underset{2}{W}^{\alpha}$ but not orthogonal to $\underset{3}{W}^{\alpha}$; $\underset{2}{W}^{\alpha}$ is orthogonal to $\underset{3}{W}^{\alpha}$.

Since $\underset{2}{W}^{pq}$ is a nonnull, as in the two previous cases, choosing corresponding frames of reference (with two degrees of freedom), one may write this vector in the form

$$\underset{2}{W}^{pq} = \underset{24}{\xi}^{pq} + i\,\underset{31}{\xi}^{pq}$$

Then for the bivectors $\underset{1}{W}$ and $\underset{3}{W}$, if the conditions of orthogonality and nullness which were indicated above are considered, we obtain the expression

$$\underset{1}{W}^{pq} = \underset{14}{\xi}^{pq} + i\,\underset{23}{\xi}^{pq} + i\left(\underset{34}{\xi}^{pq} + i\,\underset{12}{\xi}^{pq}\right)$$

$$\underset{3}{W}^{pq} = \lambda\left\{\underset{14}{\xi}^{pq} + i\,\underset{23}{\xi}^{pq} - i\left(\underset{34}{\xi}^{pq} + i\,\underset{12}{\xi}^{pq}\right)\right\}$$

where λ is some value other than zero. Further study is conducted in the same way as for the preceding type of characteristic: we write the conditions in Eq. (31) for $R_{\alpha\beta}$, bearing in mind the fact that $\underset{1}{W}^{\alpha}$ is a vector of principal direction (in bivector space), and that the vectors $\underset{1}{W}^{\alpha}$, $\underset{2}{W}^{\alpha}$, $\underset{3}{W}^{\alpha}$ determine invariant groups of the tensor $R_{\alpha\beta}$. These conditions are

$$(R_{\alpha\beta} - K g_{\alpha\beta})\,\underset{1}{W}^{\beta} = 0$$

$$(R_{\alpha\beta} - K g_{\alpha\beta})\,\underset{2}{W}^{\beta} = \sigma\, g_{\alpha\beta}\,\underset{1}{W}^{\beta} \tag{32}$$

$$(R_{\alpha\beta} - K g_{\alpha\beta})\,\underset{3}{W}^{\beta} = \tau\, g_{\alpha\beta}\,\underset{2}{W}^{\beta}$$

where σ and τ are numbers other than zero.

Considering that the bivector $\underset{\sigma}{W}^{pq}$ at a given point in T_4 determines a local metric bivector space (the vector $\underset{nt}{W}^{pq} \to \underset{\sigma}{W}^{\alpha}$) and taking coordinates such that

$$\underset{nt}{\xi}^{pq} \to \underset{\sigma}{\xi}^{\alpha} = \delta^{\alpha}_{\sigma}$$

it is easy to show that the system of equations (32) reduces to the following nine independent equations:

$$m_{11} + in_{11} + im_{13} - n_{13} = -K$$

$$m_{12} + in_{12} + im_{23} - n_{23} = 0$$

$$m_{13} + in_{13} + im_{33} - n_{33} = iK$$

$$m_{12} + in_{12} = -\sigma$$

$$m_{22} + in_{22} = -K$$

$$m_{23} + in_{23} = -\sigma_i$$

$$m_{11} + in_{11} - im_{13} + n_{13} = -K$$

$$m_{12} + in_{12} - im_{23} + n_{23} = -\tau$$

$$m_{13} + in_{13} - im_{33} + n_{33} = iK$$

where $K = \alpha + i\beta$ is one of two roots (each repeated thrice) of the secular equation

$$|R_{\alpha\beta} - Kg_{\alpha\beta}| = 0$$

and the values of σ, τ are different from zero, but otherwise arbitrary. This arbitrariness arises because of the arbitrariness of the value of λ and is a consequence of the nullness of the vectors $\underset{1}{W}^{\alpha}$, $\underset{3}{W}^{\alpha}$. One may, for example, assume that σ and τ are real numbers.

Solving this system and also taking into consideration the conditions

$$\sum_{s=1}^{3} e_s m_{ss} = \kappa$$

$$\sum_{s=1}^{3} e_s n_{ss} = 0$$

it is not difficult to see that $\tau = 2\sigma$, $\beta = 0$, $\alpha = \kappa/3$, so that the matrix $||R_{\alpha\beta}||$ takes the form

$$(R_{\alpha\beta}) = \left\| \begin{array}{ccc|ccc} -\frac{\kappa}{3} & -\sigma & 0 & .0 & 0 & 0 \\ -\sigma & -\frac{\kappa}{3} & 0 & 0 & 0 & -\sigma \\ 0 & 0 & -\frac{\kappa}{3} & 0 & -\sigma & 0 \\ \hline 0 & 0 & 0 & \frac{\kappa}{3} & \sigma & 0 \\ 0 & 0 & -\sigma & \sigma & \frac{\kappa}{3} & 0 \\ 0 & -\sigma & 0 & 0 & 0 & \frac{\kappa}{3} \end{array} \right\| \tag{33}$$

where σ is some real number other than zero; the frame of reference is determined to within a rotation in the plane $\left\{ \underset{1}{\xi} \underset{3}{\xi} \right\}$ and a Lorentz rotation in the plane $\left\{ \underset{2}{\xi} \underset{4}{\xi} \right\}$.

D. Summary

Summarizing, the following has been derived:

THEOREM 4. There exist three different principal types of gravitational field:

(1) The first type with a characteristic K-matrix of the simple type $[1\ 1\ 1, \overline{1}\ \overline{1}\ \overline{1}]$; for such a point of T_4 there is determined a real orthogonal frame of reference in which the matrix $||R_{\alpha\beta}||$ takes the form of Eq. (23) with the conditions of Eq. (24), and (25).

(2) The second type with a characteristic K-matrix of a nonsimple type $[21, \overline{2}\overline{1}]$; for it the frame of reference is determined up to two degrees of freedom, and the matrix $||R_{\alpha\beta}||$ takes the form of Eq. (29) with the conditions of Eq. (30).

(3) The third type also has a nonsimple characteristic K-matrix, of type $[3, \overline{3}]$; again the canonical frame of reference frame is determined up to two degrees of freedom, and the matrix $||R_{\alpha\beta}||$ takes the form of Eq. (33).

The three indicated types allow, of course, a further, more detailed classification. For example, one may separate the cases of multiple or real roots, as was done by us earlier.

This result, derived by the author in 1950, was first published in 1951 (Ref. 1). In that article there was an error in the formulas. The third theorem, in Part II, was also proved by A. P. Norden in 1952 (in an unpublished paper), from his research on bi-affine space. The proof used in the present work is a third variant and is, obviously, the simplest.

In regard to the work presented in Part III, that is, the determination of a canonical form of a matrix $||R_{\alpha\beta}||$ in an orthogonal rectilinear frame of reference, it is necessary to make the following remark. It might have seemed possible at once to write the canonical form of the matrix $||R_{\alpha\beta} - K g_{\alpha\beta}||$ by means of a generalization of the algebraic theory (Ref. 6). However, this is impossible to do since as coefficients of permissible linear real transformations in our present six-space, we may choose *only* an array of the form

$$A_{\alpha}^{\alpha'} = 2 A_{ij}^{[i' j']}$$

where

$$A_i^{i'} = \left(\frac{\partial x^{i'}}{\partial x^i}\right)_P$$

are the coefficients of some real orthogonal transformation at a given point P in a manifold T_4. That is, we may use only transformations that are a subgroup of the group of orthogonal real transformations of six-space.

This fact, which necessitated the arguments of Part III, is evident in the present case and is an application of a general theorem by G. B. Gurevich (Ref. 7).

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